Section 2-8, Mathematics 108

One-To-One Functions and their Inverses

What is a one-to-one function

Here is a simple example:

$f(x) = x + 4$

$$\left| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right.$$

What is not a one-to one function?

$f(x) = x^2$

$$\left| \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \right.$$

A one-to-one function is one in which each member of the range $f(x)$ has a unique x in the domain.

One way to state this is

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

The contra-positive of this statement

(Do we need to talk about what a contra-positive is?)

is $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Examples:

$f(x) = Ax + B$

Let's prove this one

Assume $f(x_1) = f(x_2)$

Then: $Ax_1 + B = Ax_2 + B$

Subtract B from both sides

$Ax_1 = Ax_2$

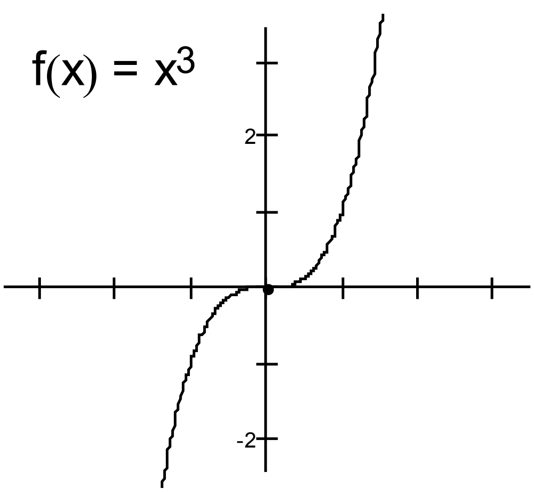
Divide both sides by A

$x_1 = x_2$

So we've proved this is a one-to-one function!

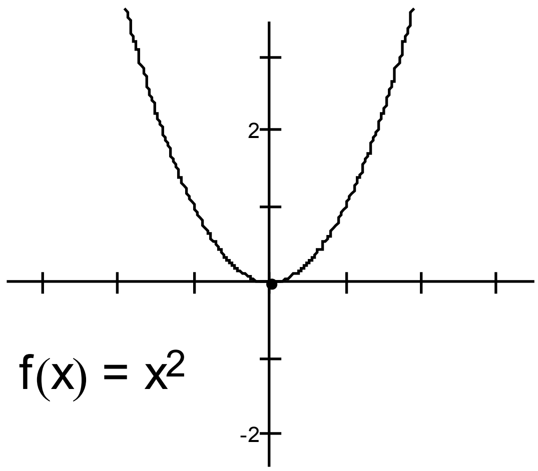
Another well known one-to-one function is

$f(x) = x^3$

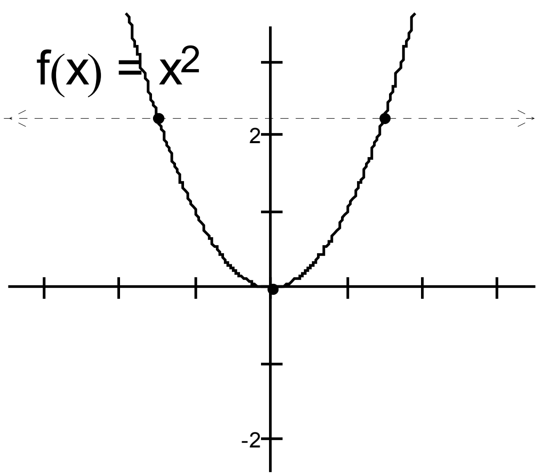


But as we saw before

$f(x) = x^2$ is not.



Is there an obvious way we can tell from this diagram?



Note that a horizontal line intersecting two points shows that this is NOT a one-to-one function.

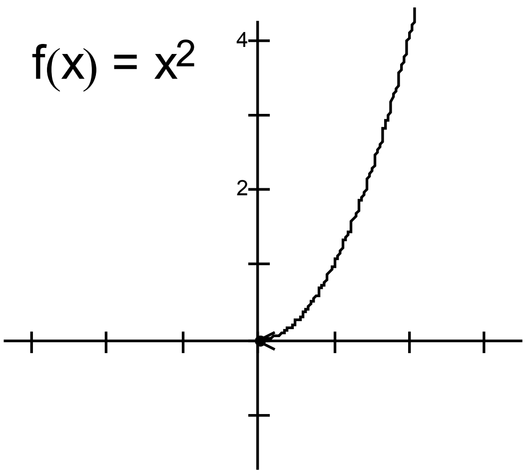
This is called the **Horizontal Line Test**.

It sometimes is possible to make a function that is not one-to-one a one-to-one function by carefully restricting it's Domain.

Here's how to do it with $f(x) = x^2$

Make the $Domain = [0, \infty)$

Now the graph looks like this:



And it passes the horizontal line test.

Inverse Functions

One very important feature of a one-to-one function is that it has an inverse.

If we have a function $f(x)$ we write it's inverse as $f^{-1}(x)$.

Do not confuse this with $\left(f(x)\right)^{-1} = \frac{1}{f(x)}$

An inverse function takes elements of the range of a function back to the element of the domain that they came from:

$$x \to f(x) \to x$$

$$\left|\begin{pmatrix}1\\2\\3\end{pmatrix} \to \begin{pmatrix}5\\6\\7\end{pmatrix} \to \begin{pmatrix}1\\2\\3\end{pmatrix}\right.$$

A formal definition of an inverse:

If $f(x)$ is one-to-one and has domain A and range B then
its inverse $f^{-1}(x)$ has domain B and range A and

$f^{-1}(y) = x$ if and only if $f(x) = y$

Example:

$f(x) = 2x + 6$

To find the inverse reverse *x* and *y* and solve for *y*.

$$y = 2x + 6$$
$$x = 2y + 6$$
$$2y = -x - 6$$
$$y = -\frac{1}{2}x - 3$$

so

$$f^{-1}(x) = -\frac{1}{2}x - 3$$

Try it out

$$f(0) = 2 \cdot 0 + 6 = 6$$
$$f^{-1}(6) = -\frac{1}{2} \cdot 6 - 6 = 0$$

Example:

$f(x) = x^2 \quad Domain = [0, \infty)$

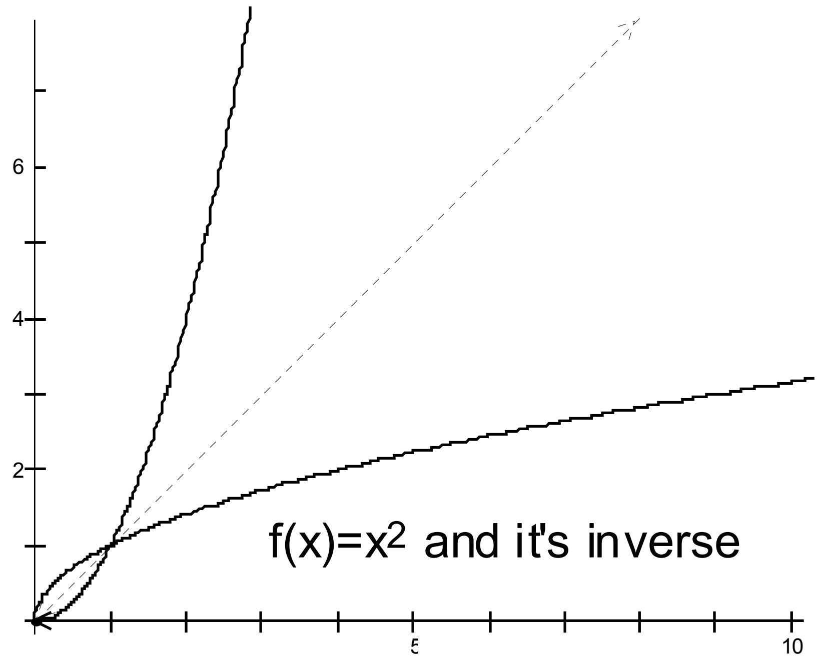
To find the inverse reverse *x* and *y* and solve for *y*.

$$y = x^2$$
$$x = y^2$$
$$y = \sqrt{x}$$

so

$$f^{-1}(x) = \sqrt{x}$$

Let's take a look at the graph of a function and it's inverse



Notice that the two functions are a reflection of each other across the line $y = x$.

If you think about how we found the inverse, by switching x and y this makes a lot of sense.

Important Properties of a function and its inverse:

If you have a function $f(x)$ and its inverse $f^{-1}(x)$

for each x in the Domain of f

$$f^{-1}(f(x)) = x$$

Also for each x in the Range of f

$$f(f^{-1}(x)) = x$$